Toward an Axiomatic Theory of Corporate Growth

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Abstract: Investment and financing decisions are generally made under the assumption of a given objective function. Deriving growth from the value of a firm, however, requires the initial size and discount rates to be known a priori. The presentation of financial planning as a standard dynamic optimization problem usually works with overly restrictive assumptions. As an alternative, it is proposed to arrive at a decision on the basis of axiomatic principles of admissibility and efficiency. Conditions for the existence of a growth program satisfying these principles are defined. The maintenance of capital is an important side constraint. Investment and financing programs, payoffs, (intrinsic) corporate value and interest rates result as endogenous magnitudes of an axiomatic model which is specified for the cases of certain and of uncertain expectations.

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1. Introduction

If managerial decisions are geared towards increasing shareholder value, a model of corporate growth must be derived from an optimization problem. Assuming the value of a share of stock to be equal to the present value of the future funds that a stockholder expects to receive from it, corporate growth is usually expressed by a function that maximizes discounted future earnings, profits, dividends, or cash flows.¹ It is only this concept of (intrinsic) value as reflecting financial (and not physical) assets of a firm that is of relevance in modeling growth.

In most of the financial economics literature, growth is related to the value of capital relative to its replacement cost (e.g., Tobin and Brainard, 1977). However, deriving growth from value involves at least two difficulties: determining the initial size of the firm, i.e. the optimal size from which to start growing at a constant rate, and determining appropriate discount factors. Opportunity costs of capital are generally not known until an optimal solution has been found. Capital market theory shows that even for publicly traded firms with a widespread shareholder base, appropriate discount factors can be derived only under very restrictive assumptions. Moreover, the majority of firms have only very limited access to capital markets. This makes such approaches overly restrictive and rather impracticable.

As an alternative, an axiomatic approach suggests itself. This method of determining growth programs attempts to restrict the realm of solutions by way of generally accepted principles (axioms) such that ultimately only one admissible solution remains. Axiomatic models have particularly been employed in welfare economics and game theory but also for measuring stagflation and unemployment, and determining price indexes (Epstein, 1986; Diewert, 1995).

The objective of this paper is to formulate and analyze such principles and to define conditions for the existence of a growth program satisfying these principles. On the basis of stock value as a measure of the growth of companies, the concepts of a growth strategy and of a growth program are introduced. Growth strategies indicate desired changes of value over the planning period whilst growth programs are characterized by the investment and financing program, distributed dividends, and interest rates necessary for determining value.

The model is formulated in a standard discrete-time, finite-space setting. All magnitudes of a growth program are taken to be unknown a priori. In this respect this approach differs from the

¹ There has been a long discussion about what should be discounted. It turns out that, properly defined, these approaches are equivalent. See Miller and Modigliani, 1961.
usual growth models formulated as optimization problems, which assume interest rates to be known. The axiomatic approach does not assume constant growth rates and therefore circumvents the problem of defining an optimal initial size at which steady states begin. It does not presuppose any specific financing program through debt or equity nor does it require the stability of dividend policy. Lastly, it does not stipulate upper bounds on growth rates or define optimal growth rates (Higgins, 1977). The model is logically consistent with Gibrat’s law of proportional effect, that firm growth is independent of firm size and that there obtains no optimal size with respect to growth. It is in keeping with the Miller-Modigliani proposition that the value of a firm is governed by its earnings and earning power (Miller and Modigliani, 1961).

The paper first introduces the determination of corporate growth as a standard dynamic optimization problem by proposing two distinct approaches to deriving conditions for optimality (Chapter 2). It then presents the alternative axiomatic model by analyzing the case of certain expectations. After a definition of growth strategies (Chapter 3.1.) and growth programs (Chapter 3.2.), two principles and five conditions are formulated for determining a growth program (Chapter 3.3.). These restrictions on optimal growth trajectories are explicated by way of an example (Chapter 3.4.). Subsequently, the consequences for corporate growth of planning the investment and financing program and payoff on the basis of an optimization exercise with a given utility function are analyzed (Chapter 3.5.). The solution proposed contains, as endogenous magnitudes of the model, not only the investment and financing program, the payoff and corporate value, but also the interest rates required for deriving the intrinsic value of a firm.

For purposes of illustration, the consequences of requiring the preservation of value (i.e. zero growth or restriction of consumption in each period to the income generated in this period) are investigated. This assumption suggests itself in models with finite planning horizons unless one wants to embrace the unrealistic assumption that there is no capital stock left after time $T$ (Takayama, 1985, p. 445f.). Lastly, the case of uncertain expectations about economic development is considered (Chapter 3.6.). A finite number of possible changes of the environment, each with a known subjective probability distribution, are allowed for. The paper ends with reflections on the relevance, extendibility, and applicability of an axiomatic theory of corporate growth (Chapter 4).

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2 For the typical assumptions of conventional growth models, see Baumol, 1962.
2. The Standard Dynamic Optimization Approach

The management of a firm sets out to maximize, by making investment decisions, the market value of the firm’s equity, \( V_{t-1} \). Under the assumption of a finite planning horizon, this amounts to the standard dynamic optimization problem expressed by

\[
\max_{\{I_s\}_{s=0}} V_{t-1} = \sum_{t=0}^{T} q_{t-1,s} d_s
\]

subject to

\[
K_s = I_s + (1-\delta)K_{s-1}
\]

where \( V \) stands for corporate value and \( d \) for the present value of future dividends discounted back to time \( \tau \) using the compound interest factor \( q \). Accordingly, management selects a time path of investment, \( \{I_s\}_{s=0}^{T} \), so as to maximize \( V \); subject to the constraint that capital stock \( K \) evolves over time dependent on the rate of physical depreciation \( \delta \). Dividends represent the capitalized stream that establishes the prevailing shareholder value. Under conditions of certainty, an individual’s time preference rate is equal to the riskless interest rate of both borrowing and lending. With maximum current stock value as an objective and time dynamics fully considered, the choice of an optimal level of retained earnings (i.e. the reciprocal of dividends) and external equity becomes the firm’s essential investment and growth problem. Optimization exercises of this type have been solved as convex optimal control problems, both in deterministic and in stochastic frameworks (Krouse and Lee 1973; Bensoussan, Hurst and Näslund 1974, ch. 4).

By the Kuhn-Tucker theorem, a solution of the optimization problem (1)-(2) is characterized by a vector of non-negative Lagrange multipliers. For each period \( s \geq t \), the Lagrangian for (1)-(2) reads

\[
L_{t-1} = \sum_{s=t}^{T} \left[ q_{t-1,s} d_s - q_{t-1,s} \pi_s \left( K_s - I_s - (1-\delta)K_{s-1} \right) \right]
\]

Hence, there is a number \( T \) of side constraints to be taken into account, namely one representing the growth of the stock of capital in each period. The term \( \pi \) in (3) denotes the period \( s \) shadow price of capital. Now, rearranging the terms inside the square brackets yields

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3 The maintenance of capital was the subject of an important debate between A.C. Pigou, F.A. Hayek and J. Hicks. See Hennings, 1990.
\[
L_{t-1} = \sum_{s=t}^{T} \left\{ q_{t-1,s} d_s - q_{t-1,s} \pi_s \left[ (K_s - K_{s-1}) - (I_s - \delta K_{s-1}) \right] \right\}
\]

(4)

and further

\[
L_{t-1} = \sum_{s=t}^{T} \left\{ q_{t-1,s} d_s - q_{t-1,s} \pi_s (I_s - \delta K_{s-1}) - q_{t-1,s} \pi_s (K_s - K_{s-1}) \right\}
\]

(5)

The first term inside the curly brackets in (5) represents period \( s \) dividends, discounted back to time \( t - 1 \), the second term denotes the increase in capital stock times the shadow price of capital \( \pi_s \), also discounted back to time \( t - 1 \). Hence, these two terms indicate the contribution to the value of the firm deriving from its period \( s \) activities. Obviously, these activities comprise, on the one hand, production and investment which, through unspecified assumptions regarding financial structure and dividend policy, yield the flow of dividends, \( d_s \). On the other hand, these activities generate the change in capital stock.

The current-value Hamiltonian can now be defined as precisely the sum of these two contributions:

\[
H_s = q_{t-1,s} d_s + q_{t-1,s} \pi_s (I_s - \delta K_{s-1})
\]

(6)

\( H_s \) shall be referred to as the flow-Hamiltonian since it expresses the contribution to the value of the firm that derives from its operations in period \( s \). By inserting (6) into (5) we can now simplify the Lagrangian and obtain

\[
L_{t-1} = \sum_{s=t}^{T} \left\{ H_s - q_{t-1,s} \pi_s (K_s - K_{s-1}) \right\}
\]

(7)

The first three terms under the summation sign, i.e. the terms relating to periods \( t, t + 1 \) and \( t + 2 \), can now be written out as follows:

\[
L_{t-1} = H_t - q_{t-1,t} \pi_t (K_t - K_{t-1})
\]

\[
+ H_{t+1} - q_{t-1,t+1} \pi_{t+1} (K_{t+1} - K_t)
\]

\[
+ H_{t+2} - q_{t-1,t+2} \pi_{t+2} (K_{t+2} - K_{t+1})
\]

\[
+ ...
\]

(8)

Notice that the capital stock in periods \( t, t + 1, t + 2 \) appears twice in the Lagrangian. Combination of these terms and collection of the Hamiltonians under the summation sign yields:
\[ L_{t-1} = q_{t-1,s} \pi_t K_{t-1} + \sum_{t=s}^{T} H_s = \sum_{t=s}^{T} \Delta(q_{t-1,s+1} \pi_{s+1}) K_s \]  
\[ + (q_{t-1,t+1} \pi_{t+1} - q_{t-1,t} \pi_t) K_t \]
\[ + q_{t-1,t+2} \pi_{t+2} - q_{t-1,t+1} \pi_{t+1}) K_{t+1} \]
\[ + \ldots \]

Considering the terms in parentheses, for some given \( K_s, s \geq t \), this implies

\[ \Delta(q_{t-1,s+1} \pi_{s+1}) = q_{t-1,s+1} \pi_{s+1} - q_{t-1,s} \pi_s \]  

The economic interpretation of (10) is that of a capital gain. It expresses the change in the shadow value of capital held through period \( s + 1 \). Multiplying this by the amount of capital available at the end of period \( s \) yields the total capital gain on the stock of productive assets held by the firm during period \( s + 1 \). Collecting these capital gains under a separate summation sign yields the simplified Lagrangian

\[ L_{t-1} = q_{t-1,s} \pi_t K_{t-1} + \sum_{t=s}^{T} H_s + \sum_{t=s}^{T} \{ \Delta(q_{t-1,s+1} \pi_{s+1}) K_s \} \]  

The firm’s management will now seek to choose investment optimally from period \( t \) onwards. Therefore the first term on the right of equation (11) does not affect optimality of decisions, since it depends solely on the stock of capital inherited from the past, \( K_{t-1} \). It can be dropped so that the Lagrangian now becomes

\[ L^*_t = \sum_{t=s}^{T} H_s + \sum_{t=s}^{T} \{ \Delta(q_{t-1,s+1} \pi_{s+1}) K_s \} \]  

According to this equation, during its growth process a firm maximizes the sum of two components. The first of these, the sum of the flow-Hamiltonians, expresses (the present value of) the return in each period that results from decisions within those periods. These decisions comprise financing and investment measures such as dividend policy, in turn reflecting productive activity and investment expenditures, and additions to the capital stock, i.e. net investment. The second component represents the present value of the capital gains on the existing capital stock in each period.

Since the Lagrangian function is differentiable, for the optimum choice of investment and capital the following first-order conditions must hold:
\[ \frac{\partial \mathcal{L}_{t-1}}{\partial l_s} = 0 \Rightarrow \frac{\partial \mathcal{H}}{\partial l_s} = 0 \] (13)

and

\[ \frac{\partial \mathcal{L}_{t-1}}{\partial K_s} = 0 \Rightarrow \frac{\partial \mathcal{H}_{s+1}}{\partial K_s} = -\Delta (q_{t-1,s+1} \pi_{s+1}) \] (14)

(13) permits one to derive Tobin’s q investment demand relation while repeated forward substitution of (14) yields shadow price \( \pi \) as the present value of the marginal product of capital.\(^4\)

First-order optimality conditions can, however, also be derived in a slightly different way. Instead of manipulating the Lagrangian and defining first derivatives of the flow-Hamiltonian, a stock-Hamiltonian may be introduced. The term in square brackets in (3) may be rearranged to read

\[ \mathcal{L}_{t-1} = \sum_{s=t}^{T} \left[ q_{t-1,s} d_s - q_{t-1,s} \pi_s [K_s - (I_s + (1-\delta)K_{s-1})] \right] \] (15)

from which one can deduce

\[ \mathcal{L}_{t-1} = \sum_{s=t}^{T} \left\{ q_{t-1,s} d_s - q_{t-1,s} \pi_s [I_s + (1-\delta)K_{s-1}] - q_{t-1,s} \pi_s K_s \right\} \] (16)

Next, the stock-Hamiltonian is introduced:

\[ \tilde{\mathcal{H}}_s = q_{t-1,s} d_s + q_{t-1,s} \pi_s [I_s + (1-\delta)K_{s-1}] \] (17)

By contrast with \( \mathcal{H}_s \), the flow-Hamiltonian in (6), the stock version \( \tilde{\mathcal{H}}_s \) has the second term represent capital stock at the end of the period multiplied by its shadow price. This is, by definition, equal to the value of the entire stock of productive capital held by the firm at the end of period \( s \). Where the flow version includes only additional capital accumulated during \( s \) and thus expresses the marginal contribution to corporate value that derives from its period \( s \) operations alone, the stock-Hamiltonian includes the dividend return in period \( s \) plus the value of the entire capital stock at the end of the period. Hence, \( \tilde{\mathcal{H}}_s \) captures the contribution to market value arising from activities beginning in period \( s \) and extending into the future. Inserting the definition of \( \tilde{\mathcal{H}}_s \) into (16) yields

\(^4\) Equations (13) and (14) provide the discrete-time analog of the necessary conditions developed, e.g., by Dorfman, 1969, and are similar to the first-order conditions derived by Dixit, 1990 (equations 10.5 and 10.11).
\[
L_{t-1} = \sum_{s=t}^{T} \left( \hat{H}_s q_{t-1,s} \pi_s K_s \right) 
\]

(18)

In contrast with (12), this equation has no readily available interpretation. Differentiation with respect to \( I_s \) and \( K_s \) yields the necessary first-order conditions:

\[
\frac{\partial L_{t-1}}{\partial I_s} = 0 \Rightarrow \frac{\partial \hat{H}_s}{\partial I_s} = 0
\]

(19)

and

\[
\frac{\partial L_{t-1}}{\partial K_s} = 0 \Rightarrow \frac{\partial \hat{H}_{s+1}}{\partial K_s} = q_{t-1,s} \pi_s
\]

(20)

Equation (20) states that the first derivative of the stock-Hamiltonian must equal the shadow price of capital in the previous period. This is quite intuitive, since differentiating \( \hat{H} \) with respect to \( K \) yields the contribution to current dividends plus the value of the capital stock at the end of the period. That is, it equals the marginal effect of additional capital on the value of the firm’s activities in periods \( s, s + 1, s + 2, \ldots \). In optimum, obviously, this contribution must be equal to the shadow price of capital at the end of the previous period.

Under both approaches, however, a certain future is assumed and the determinants of optimal growth paths have to be known \textit{a priori}. In particular, in order to solve (1)-(2) it is necessary to establish values for \( d \), which in turn presupposes that \( q \) is given for every year of the planning period. Because this assumption is rarely met, the discount rates have to be treated as endogenous variables to be determined simultaneously with the time path of investment, and with economic value. This suggests the alternative axiomatic approach.

3. The Axiomatic Approach

3.1. Growth Strategies

Earnings achieved by a firm can be either distributed to stockholders in the form of dividends or they can be reinvested in the firm.\(^5\) The value of a firm is then given by discounting the dividends

\(^5\) Whether payoffs are effected in the form of cash dividends or stock dividends (splits) is insubstantial to this model.
distributed to investors. If $d_t$ is the sum of distributed dividends, $q_t = \prod_{\tau=1}^{T}(1+i_\tau)^{-1}$ represents the discount factor at time $t$, and if $T$ is the planning horizon, value $V$ is defined by:

$$V_t = \frac{1}{q_t} \left( \sum_{\tau=t+1}^{T} d_\tau q_\tau + V_T q_T \right) = (1+i_t)V_{t-1} - d_t \quad (t = 0, \ldots, T) \quad (21)$$

Changes in value may be described by growth rates $g_t$ with $V_t = (1+g_t) \cdot V_{t-1}$ ($t = 1, \ldots, T$). A growth strategy $\mathbf{g} = (g_1, \ldots, g_T)$ is then the planned or intended development of value over the planning period.

Positive growth will be achieved by foregoing present distribution of dividends, negative growth (or contraction) will be achieved by foregoing future distributions. There is then a correlation between growth strategy $\mathbf{g}$ and distributions. Since $V_t = (1+i_t)V_{t-1} - d_t$ and $V_t = (1-g_t)V_{t-1}$, this correlation may be defined as:

$$d_t = (i_t - g_t)V_0 \prod_{\tau=1}^{t-1}(1+g_\tau) \quad (t = 1, \ldots, T) \quad (22)$$

The choice of a growth strategy therefore entails the following consequences for the distribution of dividends:

If $g_t = -1$, we have $d_t = (1+i_t)V_{t-1}$, i.e. the capitalized value of the previous period is fully ‘used up’ at $t$.

If $g_t = 0$, the corporate value remains unchanged at $t$, only interest on the previous value of the previous period (economic profit) being distributed.

If $g_t = i_t$, no distribution will occur at $t$ and corporate value at $t$ will grow by the interest accruing on the value of the previous period.

If $g_t > i_t$, a growth of value exceeding the interest accruing in the previous period is sought. Equation (1) shows that this is possible only if additional equity capital is injected at $t$.

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6 The model admits $g > i$ as a borderline case and avoids the ‘growth stock paradox’ by eschewing the unrealistic premise of an infinite horizon typical of discounting models. See Miller and Modigliani, 1961; Baumol, 1962; Levine and Zame, 1996.
3.2. Growth Programs

Whether a given growth strategy can be implemented depends primarily on the investment and planning measures taken during the planning period. In the following it will be assumed that, within any planning period $n$, investment and financing measures are available the payment relevance of which will terminate at or before the planning horizon $T$.

The surplus of inflows over outflows permitting distributed dividends at time $t$ which depend on an investment and financing program $x$ is now defined as $e_t(x) = e_t(x_1, \ldots, x_n)$. Assuming that dividends are at the earliest paid at the end of the first period and that after distribution at time $T$ there remain assets amounting to

$$V_T = V_0 \prod_{t=1}^{T} (1 + g_t),$$

we have:

$$0 = e_0(x), \quad (23)$$
$$d_t = e_t(x), \quad t = 1, \ldots, T-1, \quad (24)$$
$$d_T + V_0 \prod_{t=1}^{T} (1 + g_t) = e_T(x) \quad (25)$$

In addition to liquidity conditions (23) to (25), the investment and financing program requires further restrictions such as non-negativity or project dependencies. If these are described by set $X$, we obtain:

$$x \in X \quad (26)$$

If interest rates $i_t$ are determined exogenously, a growth program will be chosen that maximizes the initial value $V_0$ under conditions (22)-(26). However, in the following it is assumed that interest rates are unknown a priori.

What is required, then, is an investment and financing program $\bar{x}$, a vector of distributed dividends (or payoff vector) $\bar{d} = (d_1, \ldots, d_T)$, and a vector of interest rates $\bar{i} = (i_1, \ldots, i_T)$. The triple $(\bar{x}, \bar{d}, \bar{i})$ is then called a growth program.$^8$

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$^7$ Payments accruing after the planning horizon may be accommodated by discounting by exogenously determined interest rates.

$^8$ In general, programs are sequences of vectors associated with successive time periods. For a more formal definition of programs, see Koopmans, 1986.
3.3. Principles of Growth Programs

Since the discount factors necessary for determining intrinsic value are unknown \textit{a priori}, an investment and financing program permitting the implementation of a given growth strategy cannot be derived from the solution of an optimization exercise which aims at maximizing value. Hence it is proposed that, for establishing a growth program, any such program must obey two generally accepted principles.

Implementation of a growth program firstly requires compliance with liquidity conditions and other restrictions. As a first principle P1, the admissibility of the targeted growth program \((\bar{x}, \bar{d}, \bar{i})\) will be required.

**P1 (Admissibility):** \((\bar{x}, \bar{d}, \bar{i})\) fulfills conditions (22)-(26) if

\[
V_0 = \sum_{t=1}^{T} \bar{d}_t \bar{q}_t \left( \bar{q}_t = \prod_{i=1}^{t} \left( 1 + \bar{i}_t \right)^{-1} \right).
\]

In addition to the growth program being admissible, it will secondly be required that it maximizes the capital value determined by interest rates \(\bar{i}_t\). Since this requirement, in compliance with the condition of the absence of arbitrage from capital markets, guarantees the efficiency of dividend vector \(\bar{d}\) (where \(\bar{i}_t\) is the marginal profit of period \(t\)), it may be called efficiency principle.

**P2 (Efficiency):** \(x\) maximizes initial value, i.e. \(x\) is an optimal solution of

\[
\max_{x \in X} \sum_{t=0}^{T} e_t(x) \bar{q}_t, \quad \bar{q}_t = \prod_{i=1}^{t} \left( 1 + \bar{i}_t \right)^{-1}.
\]

P1 and P2 restrict conceivable growth programs. They imply the efficiency of \(\bar{d}\), where \(\bar{d}\) is efficient if there is no other admissible solution \((\hat{x}, \hat{d})\) with \((\bar{d} \geq \hat{d})\) and \((\hat{d} \neq \bar{d})\). This raises the question whether principles P1 and P2 are non-contradictory. The question can be answered in the affirmative if the following conditions are met:
C1 $e_t(x)$ is concave, $(\forall t) t = 0,\ldots, T$.

C2 $X$ is convex.

C3 $(\exists \hat{x}) \hat{x} \in X$ where $e_t(\hat{x}) > 0$, $t = 0,\ldots, T$.

C4 At every instant $t = 0,\ldots, T-1$, there is the opportunity of investing or of raising unlimited funds over a one-period term at interest $i_A$ or $i_B$, respectively.

C5 $(\forall i) i \in I$ with $i_A \leq i_t \leq i_B$ ($t = 1,\ldots, T$), $\max_{x \in X} \sum_{t=0}^{T} e_t(x)q_t$ has a finite optimal solution.

C1 admits only investments with decreasing or constant marginal revenue and financing measures with increasing or constant marginal costs. Concavity of the earnings function implies that the optimal inter-temporal trajectory is unique and converges toward a steady state. C2 rules out particularly that all $x$ have to be integers or that fixed cost discontinuities are admissible. C3 corresponds to the non-negativity condition (or Slater’s condition) known from the theory of non-linear programming (Takayama 1985, p. 73). C5 guarantees that marginal profits are finite for every efficient payoff vector.

Under conditions C1 to C5, principles P1 and P2 are tantamount to the requirement of efficient payoff vectors. If P1 and P2 are fulfilled, $(x,\overline{d},\overline{I})$ preserves the maximum intrinsic value to be attained under these conditions. At the beginning of each time period $1 \leq t \leq T$, an amount of $e_{t-1}$ is invested and transformed into capital which can be consumed or saved for the next period $t+1$. This can be interpreted as an inter-generational allocation problem.

Conditions C1 to C5 guarantee that for every environment, i.e. for every given $x$ and $i_t$, desirable growth trajectories are picked from the admissible and efficient set by a choice function. Violation of some of these conditions does not require total abandonment of the model presented. Appropriate modifications may particularly accommodate integer conditions. If C4 is not fulfilled because unlimited borrowing at interest rate $i_B$ is not possible, $i_B$ must be increased so that utilization of this financing tool is excluded a priori.9

Disregarding the unrealistic case of a perfect (or complete) capital market, marginal profits are unknown a priori. At high growth rates it thus cannot be excluded that some growth rates will be higher than the corresponding marginal profits and some payoffs will therefore be negative. In this

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9 For an extension of the basic model to accommodate the case of limited borrowing, see Bensoussan, Hurst and Näslund 1974, 76f.; see also Levine and Zame, 1996.
case the growth strategy can only be implemented if equity capital is added at a later stage. If this is excluded on a priori grounds, this has to be accounted for in condition (26) or else growth rates have to be lowered until all payoffs are non-negative (Higgins 1977).

3.4. An Illustration

For the case of certain expectations, the model may be further explicated by using the following assumptions. An initial capital of 100 units is to be invested over two periods by preserving its intrinsic value. At \( t = 0 \), the only option is a one-period (unlimited) investment at \( i = 10\% \). The efficient payoffs in \( t = 1 \) and \( t = 2 \) are then as depicted in Figure 1.

Since investment in the first period is possible only at \( i = 10\% \), marginal profit in this period is \( \bar{r}_i = 10\% \). At a given interest \( i_2 \) for the second period, efficient payoffs \( d_1^* \) and \( d_2^* \) are given at \( B \), i.e. at the intersection of line \( \varphi \) (with the slope \((1 + i_2)/0.1\)) with the payoff possibility curve \( d_1d_2 \). \( \varphi = \varphi (d_1^*, d_2^*) \) may be understood to be a payoff vector. \( d_1^* \) and \( d_2^* \) are then the maximum payoffs possible if value is to be preserved and if interest rates of periods one and two are, respectively, 10\% and \( r_2 \).

\[ \text{slope } \varphi = \frac{1 + i_2}{0.1} \]

*Figure 1: Efficient payoffs in a two-period model*
If $q^*$ is the slope of the tangent to the payoff possibility curve at $B$, $-(q^*+1)$ will be the marginal profit in period two. If $i_2$ is varied, this amounts to a rotation of $\varphi$ with marginal profit decreasing as $i_2$ increases. If $F(i_2)$ designates the set of marginal profits possible at $i_2$ ($F$ being a correspondence, since at any point $A$ there are several marginal profit rates), Figure 1 shows the existence of an interest rate $\bar{i}_2$ with $\bar{i}_2 \in F(i_2)$. $i_2$ is the fixed point of correspondence $F$ and is thus the required marginal profit of period two.

In standard models of the optimal investment decision, the net present value (NPV) corresponding to $B$ is equal to

$$NPV = \frac{d_2^*}{1+i} - (110 - d_1^*) = \left(\frac{d_2^*}{1+i} + d_1^* - 110\right)$$

(27)

Notice that since by investing $110 - d_1^*$ units one has invested up to the point at which $(d_1,d_2)' = q^*$, for the optimal investment decision it must also hold that the NPV of marginal investment equals zero. From (27) we can deduce that there is an infinite number of combinations of $d_1^*$ and $d_2^*$ all leading to the same value of NPV. In this way iso-NPV lines can be drawn in the $d_1d_2$ plane, the highest NPV, represented by $q^*$, being reached by investing $110 - d_1^*$.

Generalization of this example leads to the following parametric optimization exercise $R(i)$ with respect to interest rates $i_t (t = 1, \ldots, T)$:

$$\max_{x \in X} \left\{ V \mid 0 = e_0(x), i_t V = e_i(x) \text{ for } t = 1, \ldots, T-1, (1 + i_T) V = e_T(x) \right\}$$

(28)

If $F(i)$ stands for the set of endogenous interest vectors (marginal profits) of $R(i)$, this example illustrates that the existence of a solution that preserves intrinsic value amounts to the question of the existence of a fixed point of correspondence $F(i)$ for a suitable area over which $i$ is defined.

### 3.5. Retrospective Determination of Growth Rates

An optimal investment and financing program is usually derived from an optimization exercise

$$\max \left\{ U(d_1, \ldots, d_{T-1}, d_T^*) \mid (23) - (26) \right\}$$

(29)

on the basis of a utility function $U(d_1, \ldots, d_{T-1}, d_T^*)$, where $d_T^* = d_T + V_T$.

$U$ is assumed to be strictly concave and monotonically increasing, i.e. $U(\cdot)' > 0$ and $U(\cdot)'' < 0$. Application of the Kuhn-Tucker conditions under requirements C1 to C5 will then prove the exis-
existence of marginal profits (endogenous interest rates) that permit the definition of an optimal program by the method of net present value.

If \((\bar{x}_1, \ldots, \bar{x}_n, \bar{d}_1, \ldots, \bar{d}_{T-1}, \bar{d}_T^*)\) is an optimal solution of (28) and \(\bar{I}\) is a vector of marginal profits, the foregoing reasoning permits to determine whether and to which extent pursuing this solution implies corporate growth. In order to do so, intrinsic value

\[ \bar{V}_0 = \sum_{t=1}^{T-1} \bar{d}_t \bar{q}_t + \bar{d}_T \bar{q}_T \]

is first computed. Equation (22) then permits successive derivation of \(g_1\) to \(g_T\) from

\[ g_t = \bar{i}_t - \bar{d}_t \left( \frac{\bar{V}_0 \prod_{\tau=1}^{t-1} (1 + g_\tau)}{\bar{V}_0} \right) \]  \( \text{(30)} \)

It is then possible to determine, for every period within the planning horizon, whether implementation of the optimal program leads to an increase or decrease of value or keeps value constant. This, then, permits to determine to which extent pursuit of certain objectives is compatible with a targeted growth strategy.

### 3.6. Growth Strategies under Uncertainty

The foregoing considerations can be extended to the case of uncertain expectations. For this purpose it is assumed that the business environment is described by an event-tree with a finite number of states (Magill and Quinzii 1996: ch. 4). \(S_t\) is assumed to denote the set of states at time \(t\) \((S_0 = \{0\})\), and \(p_s\) is the subjective probability for \(s\) to occur where \(p_s > 0\). \(e_t(x)\) is the payments surplus at time \(t\), the vector \(x\) being composed of the activity levels of all measures feasible at times \(t = 0, \ldots, T-1\).

We then postulate a known probability distribution \(p(S_t)\) for the value of the firm at the end of the period, where \(S\) is a set of random variables defined by:

\[ S = \frac{V_{t+\tau}}{V_t} \]
$S$ reflects both the uncertainty about the cash flow (or earnings) of the firm and the changes in value of the firm's capital stock or earning assets. We assume again that $p(S_t)$ is independent of the particular capital structure of the firm.

At every point of time, then, several alternative developments of the business environment have to be anticipated for the subsequent period. For each of these alternatives, an interest rate $i_s$ is assumed, $s$ being one of the possible states at the end of the period. The alternatives can be presented in the form of a tree with two states at $t = 1$ and five states at $t = 2$ (Figure 2):

![Event-tree](image)

Figure 2: Event-tree

Extending the case of certain expectations, vector $\mathbf{x}$ contains the activity levels of all investment and financing measures admissible in states $s \in S_t (t = 0, ..., T-1)$, and vector $\mathbf{d}$ describes the dividends distributed in all states $s \in S_t (t = 1, ..., T)$.

A growth strategy $\mathbf{g}$ expresses how value changes as a result of all possible changes in the business environment. Consequently, every period may have as many changes in value as there are alternative developments of the business environment (Figure 2). Graphically, growth rates are indexed in accordance with interest rates. This permits to adapt a growth strategy to a forecast of business development.

If $s \in S_t$ is a state feasible at $t$ and $s_0, ..., s_t$ is the line leading to this state from state $s_0 = 0$, choice of growth strategy $\mathbf{g}$ will yield value
\[ V_s = V_0 \prod_{r=1}^{t} (1 + w_{s_r}) \ (s_r := s) \]

By analogy with the case of certain expectations, the following correlation between payoffs and growth strategy \( g \) obtains:

\[ d_s = (i_s - g_s)V_0 \prod_{r=1}^{t-1} (1 + g_{s_r}) , \ (\forall s) \ s \in S_r \ , \ t = 1, \ldots, T \] (31)

By analogy with (23)-(26), the following conditions are to be observed:

\[ d_s = e_s(x) , \ (\forall s) \ s \in S_r \ , \ t = 1, \ldots, T-1 , \] (32)

\[ d_s + V_0 \prod_{r=1}^{T} (1 + g_{s_r}) = e_s(x) , \ (\forall s) \ s \in S_T , \] (33)

\[ 0 = e_0(x) , \] (34)

\[ x \in X . \] (35)

By analogy with P1 and P2, observance of principles P1’ and P2’ is required for the targeted growth program \((\bar{x}, \bar{d}, \bar{i})\):

**P1’ (Admissibility):** \((\bar{x}, \bar{d}, \bar{i})\) is admissible, i.e. it fulfills conditions (31) to (35).

**P2’ (Efficiency):** \( \bar{x} \) maximizes the expected value of initial value \( V_0 \) , i.e. \( \bar{x} \) is an optimal solution of

\[
\max_{x \in X} \sum_{r=0}^{T} \sum_{s \in S_r} e_s(x)p_s \bar{q}_s \quad \text{with} \quad \bar{q}_s = \prod_{r=1}^{t} (1 + i_{s_r})^{-1} \quad (s_r := s) . \tag{10}
\]

\( \bar{q}_s \) is the discount factor to be assumed for \( s \in S_r \) and \( \sum_{s \in S_r} e_s(x)p_s \bar{q}_s \) is the NPV of payment surpluses at \( t \).
As in the case of certain expectations, a solution of conditions P1’ and P2’ derives from a solution of (32) - (35), which amount to a parametric optimization exercise. If \( F(i) \) designates the set of endogenous interest vectors of \( R'(i) \), the existence of a fixed point can be inferred and the existence of a growth program satisfying conditions P1’ and P2’ can be proved if the following conditions hold:

\[
C1' \quad e_t(x) \text{ is concave}, \quad (\forall s) s \in S_t, \quad t = 0, \ldots, T.
\]

\[
C2' \quad X \text{ is convex}.
\]

\[
C3' \quad (\exists \hat{x}) \hat{x} \in X \text{ where } e_t(\hat{x}) > 0, \quad (\forall s) s \in S_t, \quad t = 0, \ldots, T.
\]

\[
C4' \quad \text{In every state } s \in S_t, \quad (t = 0, \ldots, T-1), \text{ unlimited funds can be invested over a one-period term at interest rate } i^*_t \geq 0. \text{ In every state } s \in S_t, (t = 0, \ldots, T-1) \text{ and for every immediately subsequent state } s^+, \text{ there are furthermore conditional one-period contracts at an interest rate of } i^*_t. \text{ These include the possibility of raising unlimited funds in state } s \text{ with the proviso that obligation to pay interest and amortization will not arise before } s^+ \text{ has occurred.}
\]

\[
C5' \quad (\forall i) i \in I' \text{ and } s \in S_t, \quad (t = 1, \ldots, T) \text{ with } (1 + i^*_s)^t p(s | s^*) - 1 \leq s^*_t \leq (1 + i^*_s)^t p(s | s^*) - 1,
\]

\[
\max_{x \in X} \sum_{t=0}^{T} \sum_{s \in S_t} e_s(x)p_sq_s \text{ has a finite optimal solution (where } p(s | s^*) \text{ is the probability of state } s \text{ occurring if immediately antecedent to it state } s^+ \text{ has occurred).}
\]

As in the case of certain expectations, P1’ and P2’ amount to the condition of efficient payoff vectors, which is formally equivalent to requiring the absence of arbitrage from capital markets. But there is a qualitative difference in that the case here discussed of efficient payoff includes also decisions on real investments, for which there usually are no markets comparable to capital markets.

P1’ and P2’ do not exclude negative payoff. In the case of an unfavorable development of the business environment, particularly if \( g_s > F_s \) or \( F_s < 0 \), there will be negative dividends and additional equity capital will have to be injected for implementing the growth strategy. If, however,
this is excluded, it has to be taken account of in formulating restriction (35) or else the targeted growth rates must be sufficiently reduced.

Whereas C5 is practically always fulfilled, since it only requires interest vectors to have non-negative components, C5’ admits also negative components for interest vectors in I’. This may lead to R’(i) having an unbounded solution for some interest vectors in I’. If, on a priori grounds, an upper bound \( V_{\text{max}} \) can be defined for the intrinsic value, this may be excluded by requiring that R’(i) be solved under the additional condition of \( V \leq V_{\text{max}} \).

Under conditions C1’, C2’, C3’ and C5’, we may attribute opportunity costs \( K_s = K_s(i) \) to every liquidity condition of \( R(i) \)’ and every state \( s \). If \( s’ \) stands for the immediate predecessor state of \( s \) and \( N(s+) \) for the set of immediate successor states \( s’ \) of \( s \), the financial measures of C4’ will, for all \( s’ \in S, t = 1, \ldots, T \), postulate the dual restrictions:

\[
K_{s^+} \geq (1 + i^A_s) \sum_{s \in N(s^+)} K_s \tag{36}
\]

\[
K_{s^-} \leq (1 + i^B_s)K_s, (\forall s \in N(s^+) \) \tag{37}
\]

If \( K_s \) is rewritten as \( K_s = p_s u_s, u_s \) may be interpreted as the realization of a state variable \( U_s \) occurring with probability \( p_s \), and

\[
j_s := \frac{u_s^-}{u_s} - 1 = \frac{K_s^-}{K_s} p(s|s^-) - 1, s \in S, \tag{38}
\]

as the realization of the endogenous interest rate of period \( t \) (\( t = 1, \ldots, T \)) occurring with probability \( p(s|s^-) \).

With regard to \( j_s \), inequalities (36)-(37) permit the derivation of the estimate

\[
(1 + i^A_s) p(s|s^-) - 1 \leq j_s \leq (1 + i^B_s) p(s|s^-) - 1, (\forall s) s \in S, \tag{39}
\]

which corresponds structurally to C5’. The financing condition expressed by C4’ therefore guarantees that, for all \( i \in I’ \), the vector of endogenous interest rates \( R’(i) \) lies within \( I’ \), i.e. that all \( i \) lie within the realm over which (39) is defined. This permits the application of Kakutani’s Fixed Point Theorem (Takayama 1985, p. 259f.). The theorem demonstrates the sufficiency of the conditions for a solution (or of competitive general equilibrium) to exist.\(^{11}\)

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\(^{11}\) Restriction to a finite planning horizon permits P1’, P2’ and C1’ – C5’ to rule out Ponzi schemes (doubling strategies) and makes the assumption of the existence of equilibrium unproblematic. Unlimited amounts of borrowing are
4. Conclusion and Application

The literature on investment and financing usually postulates an optimal program on the basis of a given objective function. Since such a function may be found only very rarely, this paper takes the alternative approach of modeling corporate growth on an axiomatic basis. By foregoing several assumptions which have become standard in the literature on finance but which are often unrealistic, it seeks to develop a ‘pure’ theory of corporate growth predicated only on uncontroversial axioms. The axiomatic theory is not an economic model in the sense of being based on behavioral parameters of economic agents.

The model relies on the principles of the admissibility of the targeted growth program and the efficiency of the payoff vector. It yields not only an investment and financing program in keeping with certain optimality conditions but also, as endogenous magnitudes, the distributed dividends and the interest rates necessary for establishing the value of a firm.

The approach proposed avoids the problems inherent in standard dynamic optimization models of optimal growth paths. It is in keeping with the Miller-Modigliani thesis of the irrelevance of dividend policy if this thesis is, in its original formulation, restricted to the price of shares or the total return to shareholders in perfect capital markets (Miller and Modigliani 1961, p. 414). It has been shown that the theorem only rules out cases of firms allowing dividend decisions to affect investment decisions. In the present model, the investment and financing program is independent of the payoff vector, and a firm’s capital structure has no effect on its value. Corporate value and its growth are determined solely by the characteristics of the asset side of the balance sheet and are not affected by the particular instruments used by the firm to finance these assets.

Appropriate state-dependent discount rates can be found without assuming complete markets for future payoffs. A consistent integration of the principle of maintaining capital intact into utility maximization models is therefore possible. This result may be employed in the current debate about sustainable growth and inter-generational consumption levels of depletable resources. Macroeconomic growth theory understands a balanced growth path as being one along which all real quantities grow at constant, though not necessarily identical, rates. Traditional capital budgeting

impossible. However, the constraints expressed in the principles and conditions may be formalized in various alternative ways which lead to equivalent notions of equilibrium (Levine and Zane, 1996; Magill and Quinzii, 1996).
models usually add a further restriction by requiring that a balanced (or sustainable) growth rate be the highest growth rate a firm can maintain without increasing its financial leverage. Sustainability then requires the possibility of expansion by preserving the structure of a system, a condition that has been captured by P2 and C5 in the axiomatic model even without the assumption of steady states. Preservation of value - or of capital not in the sense of capital goods but of net wealth - seems to guarantee stability of growth trajectories (Nicola, 1994). This conclusion reminds one of Hicks’ definition of “a man’s income as the maximum value he can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning” (Hicks 1946, p. 172). Remaining equally well off means maintaining capital intact, so growth according to Hicks would by definition be sustainable (Hennings, 1990).

This idea also finds application in the discussion on corporate governance, where the legal obligation to preserve capital and to restrict dividends to the income created by a stock corporation, as it exists in several European countries and is required under EU law, is being reconsidered as to its more general merits (Monks and Minow, 1995).12

The solution of the model depends solely on the opportunities available to a firm, and on developments in the business environment. The explicit stipulation of a risk utility function is not necessary. But acting in the interest of maximizing corporate growth presupposes homogeneity of expectations or at least a consensus on the feasibility and likelihood of developments in the business environment.

The axiomatic model admits of multiple extensions. Integer conditions, investment and financing measures exceeding the planning horizon, or planning in the presence of imperfectly known probabilities, can all be accommodated within the axiomatic framework. This approach also allows for its integration into microeconomic decision models in an economy with incomplete markets and for its application to macroeconomic planning problems, where prices for future consumption are usually not given (Magill and Quinzii, 1996). The model permits various applications in dynamical portfolio management, particularly where a value-maintaining funds performance is required. It can thus be fully embedded into the capital-market theory of finance.

12 Thus the German Act on Limited Liability Companies stipulates that funds necessary to preserve paid-up capital must not be disbursed to shareholders (§ 30 GmbHG; similarly the Austrian Act on Limited Liability Companies, §§ 7 and 82 GmbHHG). The same Act requires directors to call a shareholders’ meeting if half of paid-up capital has been lost (§ 49 para. 3 GmbHG; similarly the Austrian Act, § 36 para. 2 GmbHG, and the French Act on Commercial Companies, § 241 Loi no. 66-537). The Second Council Directive, 77/91/EEC of 13 December 1976, amended by Council Directive
92/101/EEC of 23 November 1992, require of public limited liability companies in EU member countries several measures aimed at maintaining capital for the purpose of shareholder and creditor protection.
REFERENCES


